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MAPPINGS OF BOUNDED DILATATION OF RIEMANNIAN MANIFOLDS

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1. Introduction

Let M and N be Riemannian manifolds of dimensions m and n, respectively. Recently, two of the authors introduced the concept of a quasiconformal mapping $f: M \to N$ and applied it to obtain distance and (intermediate) volume decreasing properties of harmonic mappings between Riemannian manifolds of different dimensions [2], [3]. In this paper the concept of a mapping $f: M \to N$ of bounded dilatation is introduced which is more general and natural than that of a K-quasiconformal mapping when m and n are greater than 2. An example of such a mapping which is not K-quasiconformal is given which is even harmonic. In § 5, generalizations of the Schwarz-Ahlfors lemma as well as Liouville's theorem and the little Picard theorem are given for this class of mappings.

Let $f: M \to N$ be a harmonic mapping of bounded dilatation of Riemannian manifolds. If the upper bound $\|f_*\|^2$ of the ratio of distances attains a maximum at $x \in M$, then under suitable conditions on the bounds of the sectional curvatures at x and $f(x)$, f is distance decreasing.

If M is a complete connected Riemannian manifold of constant negative curvature $-A$, in particular, if M is the unit open m-ball with the hyperbolic metric of constant curvature $-A$, then the condition on $\|f_*\|$ may be dropped by virtue of the technique employed in § 5. Indeed, let N be a Riemannian manifold with sectional curvatures bounded above by a negative constant depending on A. Then, if $f: M \to N$ is a harmonic mapping of bounded dilatation, it is distance decreasing.

The technique employed to prove this statement also yields the following fact.

Let M be a complete connected locally flat Riemannian manifold and let N be an n-dimensional Riemannian manifold with negative sectional curvature bounded away from zero. Then, if $f: M \to N$ is a harmonic mapping of bounded dilatation, it is a constant mapping.

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2. Mappings of bounded dilatation

Let V be a Euclidean vector space of dimension m and let V^* be its dual space. Let $\{e_1, \cdots, e_m\}$ be an orthonormal basis of V with dual basis $\{\omega_1, \cdots, \omega_m\}$. A quadratic function on V is an element of $(V \otimes V)^*$, so since $(V \otimes V)^*$ is canonically isomorphic to $V^* \otimes V^*$, a quadratic function on V may be written as $f = \Sigma f_{ij}\omega_i \otimes \omega_j$. If f is symmetric and positive semidefinite an orthonormal basis $\{e_i\}$ can be chosen so that $f_{ij} = 0$ for $i \neq j$ and $f_{ii} = \gamma_i^2 > 0$ for $i = 1, \cdots, k \leq m$, where $k = \text{rank } f$.

Let W be a Euclidean vector space of dimension n with inner product h, and let $F: V \to W$ be a linear mapping of rank $k \leq \min(m, n)$. We choose an orthonormal basis $\{e_i\}$ of V so that

$$F^*h = \sum \gamma_i^2 \omega_i \otimes \omega_i .$$

The vectors $\eta_i = (1/\gamma_i)Fe_i$, $i = 1, \cdots, k$, form part of an orthonormal basis of W. (If all of the γ_i vanish, $F = 0$.) Let $X = \Sigma_1^m x^i e_i$ be a vector of unit length and assume $F \neq 0$; then $FX = \Sigma y^i \eta_i$, where $x^i = y^i/\gamma_i$. Consequently, if F is of rank k, it maps a unit $(k-1)$-dimensional sphere of V to a $(k-1)$-dimensional ellipsoid of W with semiaxes of lengths $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_k > 0$, where $\gamma_i^2 = \lambda_i$, $i = 1, \cdots, k$, are the eigenvalues of ${}^tFF: V \to V$.

Definition 1. The ratio

$$l_s = \gamma_1/\gamma_{s+1} , \qquad s = 1, \cdots, k-1$$

will be called the s-th dilatation of F.

The mapping $F: V \to W$ induces a mapping $\bigwedge^p F: \bigwedge^p V \to \bigwedge^p W$, $p \leq \min(m, n)$ given by

$$\bigwedge^p F(e_{i_1} \wedge \cdots \wedge e_{i_p}) = Fe_{i_1} \wedge \cdots \wedge Fe_{i_p} ,$$

where $1 \leq i_1 < i_2 < \cdots < i_p \leq \min(m, n)$. We define the norm $\| \bigwedge^p F \|$ by

$$\| \bigwedge^p F \|^2 = \sum_{i_1 < \cdots < i_p} \langle \bigwedge^p F(e_{i_1} \wedge \cdots \wedge e_{i_p}), \bigwedge^p F(e_{i_1} \wedge \cdots \wedge e_{i_p}) \rangle .$$

Thus

$$\| \bigwedge^p F \|^2 = \sum_{i_1 < \cdots < i_p} \lambda_{i_1} \cdots \lambda_{i_p} .$$

If $1 \leq p \leq q \leq s < k$ and $l_s \leq K$, the following fact is easily established.

Lemma 2.1.

$$\left[\frac{\| \bigwedge^p F \|^2}{\binom{k}{p}} \right]^{1/p} \leq K^2 \left[\frac{\| \bigwedge^q F \|^2}{\binom{s}{q}} \right]^{1/q} .$$

We shall require an inequality reversing that in Lemma 2.1. We put $\mu_0 = 1$ and $\mu_p = \Sigma\lambda_{i_1}\cdots\lambda_{i_p}/\binom{k}{p}$, $1 \le i_1 < \cdots < i_p \le k$. Since $\lambda_i \ge 0$, by Newton's inequalities we have $\mu_{p-1}\mu_{p+1} \le \mu_p^2$ and therefore $\mu_1 \ge \mu_2^{1/2} \ge \cdots \ge \mu_k^{1/k}$. These inequalities imply

$$(2.1) \qquad \left[\frac{\|\wedge^p F\|^2}{\binom{k}{p}}\right]^{1/p} \ge \left[\frac{\|\wedge^q F\|^2}{\binom{k}{q}}\right]^{1/q}, \qquad 1 \le p \le q \le k.$$

In the sequel, it is assumed that M and N are Riemannian manifolds of dimensions m and n, respectively. Let $f: M \to N$ be a C^∞ mapping, and $(f_*)_x: T_x(M) \to T_{f(x)}(N)$ be the induced mapping of tangent spaces at x.

Definition 2. If either $(f_*)_x = 0$ at each point $x \in M$ or any one of the dilatations $l_i(x)$, $i = 1, \cdots, k-1$, is bounded on M, then f is said to be of bounded dilatation. For a nonconstant mapping of bounded dilatation, $l_1(x)$ is always bounded. In this case, K will denote the *l.u.b.* of $l_1(x)$ and f will be said to be of bounded dilation of order K.

Remark. Since $l_i(x) \le l_j(x)$ for $i \le j \le k$, a K-quasiconformal mapping in the sense of [2] and [4] is a mapping of bounded dilatation. If $m = n = 2$ the two notions are identical. However, for m and n greater than 2, a mapping of bounded dilatation is not necessarily quasiconformal as the following example shows.

Let U be the open submanifold of E^3 given by $\{(x, y, z) \in E^3 \mid x^2 + y^2 > 1/(a+1)^2, a \ne -1\}$ and let $f: U \to E^3$ be defined by

$$f = \left(\frac{1}{2}(x^2 - y^2), 3xy, \frac{1}{a+1}z\right).$$

Then the eigenvalues of ${}^t f_* f_*$ are $\lambda_1 = 9(x^2 + y^2)$, $\lambda_2 = x^2 + y^2$ and $\lambda_3 = 1/(a+1)^2$. Consequently $l_1(x, y, z) = 3$ and $l_2(x, y, z) = 3(a+1)(x^2+y^2)^{1/2}$. Observe that f is also harmonic (see § 3).

In the sequel, a mapping of bounded dilatation will be assumed to have the same rank k at each point of M.

Lemma 2.2. *A C^∞ mapping $f: M \to N$ is of bounded dilatation of order K if and only if*

$$\|f_*\|^2 \le k\, K^2 \|\wedge^2 f_*\|.$$

Proof. The necessity follows from Lemma 2.1. For the sufficiency suppose that $l_1 = (\lambda_1/\lambda_2)^{1/2}$ is unbounded. Then

$$\frac{\|f_*\|^2}{\|\wedge^2 f_*\|} = \frac{\sum \lambda_i}{\left(\sum_{i<j} \lambda_i\lambda_j\right)^{1/2}}$$

$$= \left(\frac{\lambda_1}{\lambda_2} + 1 + \frac{\lambda_3}{\lambda_2} + \cdots + \frac{\lambda_k}{\lambda_2}\right) \Big/ \left(\frac{\lambda_1}{\lambda_2} + \text{terms} \leq \frac{\lambda_1}{\lambda_2}\right)^{1/2}$$

$$\geq \frac{\lambda_1}{\lambda_2} \Big/ \left[\binom{k}{2}^{1/2} \left(\frac{\lambda_1}{\lambda_2}\right)^{1/2}\right] = \left(\frac{\lambda_1}{\lambda_2}\right)^{1/2} \Big/ \binom{k}{2}^{1/2} = l_1 \Big/ \binom{k}{2}^{1/2},$$

so $\|f_*\|^2/\|\wedge^2 f_*\|$ is unbounded.

3. Harmonic mappings

In this section, the conditions for a harmonic mapping f and a formula for the Laplacian of $\|f_*\|^2$ are given. By the method of moving frames we write, locally, the metric ds^2 of a Riemannian manifold M of dimension m as

$$ds^2 = \omega_1^2 + \cdots + \omega_m^2 ,$$

where the ω_i are linear differential forms in M. The structure equations are

$$d\omega_i = \sum_j \omega_j \wedge \omega_{ji} , \qquad \omega_{ij} + \omega_{ji} = 0 ,$$
$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} + \Omega_{ij} , \qquad \Omega_{ij} + \Omega_{ji} = 0 ,$$

where the ω_{ij} are the connection forms and the Ω_{ij} are the curvature forms. If $\{e_i\}$ is the orthonormal frame dual to the coframe $\{\omega_j\}$, the connection D in the tangent bundle is given by

$$De_i = \sum_j \omega_{ij} e_j .$$

The Ω_{ij} may be expressed as

$$\Omega_{ij} = -\tfrac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l ,$$

where the functions R_{ijkl} are the components of the curvature tensor. The Ricci tensor R_{ij} is defined by

$$R_{ij} = \sum_k R_{ikjk}$$

and the scalar curvature R by

$$R = \sum_i R_{ii} .$$

Let N be a Riemannian manifold of dimension n (not necessarily that of M) and let $f: M \to N$ be a C^∞ mapping. Corresponding quantities in N will be denoted with an asterisk. Thus the Riemannian metric ds^{*2} of N is given by $ds^{*2} = \Sigma\omega_a^{*2}$. (In the sequel, we will use the convention $i, j, k, \cdots = 1, \cdots, m$

and $a, b, c, \cdots = 1, \cdots, n$.) Under the mapping f a tensor field with components A_i^a is defined by

$$f^*\omega_a^* = \sum_i A_i^a \omega_i \ . \tag{3.1}$$

Later on we will drop f^* in such formulas when its presence is clear from context. Taking the exterior derivative of (3.1) and using the structure equations in M and N, we get

$$\sum_i DA_i^a \wedge \omega_i = 0 \ ,$$

where

$$DA_i^a = dA_i^a + \sum_k A_k^a \omega_{ki} + \sum_c A_i^c \omega_{ca}^* = \sum_j A_{ij}^a \omega_j \quad \text{(say)} \ , \tag{3.2}$$
$$A_{ij}^a = A_{ji}^a \ .$$

The mapping f is said to be harmonic if

$$\sum_i A_{ii}^a = 0 \ .$$

The simplest case is a smooth mapping $f = (f_1, \cdots, f_n) : E^m \to E^n$. Then $f_* = \Sigma A_i^a dx_i \otimes \partial/\partial y_a$, where x_i and y_a are the coordinates in E^m and E^n respectively and $A_i^a = \partial f_a/\partial x_i$. Hence

$$Df_* = \sum_{a,i,j} A_{ij}^a dx_i \otimes dx_j \otimes \partial/\partial y_a \ ,$$

where $A_{ij}^a = \partial^2 f_a/\partial x_j \partial x_i$. Classically, f is harmonic if and only if

$$\sum_i A_{ii}^a = \sum \frac{\partial^2 f_a}{\partial x_i^2} = 0 \ , \qquad a = 1, \cdots, n.$$

Differentiating (3.2) and using the structure equations in M and N, we get

$$\sum_j DA_{ij}^a \wedge \omega_j = \sum_j A_j^a \Omega_{ji} + \sum_b A_i^b \Omega_{ba}^* \ ,$$

where

$$\begin{aligned} DA_{ij}^a &= dA_{ij}^a + \sum_k A_{kj}^a \omega_{ki} + \sum_k A_{ik}^a \omega_{kj} + \sum_b A_{ij}^b \omega_{ba}^* \\ &= \sum_k A_{ijk}^a \omega_k \quad \text{(say)} \ . \end{aligned} \tag{3.3}$$

For a C^∞ function φ on M the Laplacian $\Delta\varphi$ is defined in terms of the covariant differential ∇ in M by

$$\Delta\varphi = \sum_k \nabla^2 \varphi(e_k, e_k) \ .$$

Applying this definition to $\varphi = \|f_*\|^2 = \langle \Sigma A_i^a \omega_i \otimes e_a^*, \Sigma A_i^a \omega_i \otimes e_a^* \rangle$ and using the Leibnitz rule, we have

$$\nabla\varphi = 2\langle \sum_{a,i} DA_i^a \omega_i \otimes e_a^*, \sum_{a,i} A_i^a \omega_i \otimes e_a^* \rangle = 2 \sum_{a,i} A_i^a DA_i^a ,$$

$$\nabla^2\varphi = 2 \sum_{a,i} (DA_i^a DA_i^a + A_i^a D^2 A_i^a) ,$$

the latter becoming, by (3.2) and (3.3),

$$\nabla^2 \|f_*\|^2 = 2 \sum_{a,i,j,k} (A_{ij}^a A_{ik}^a + A_i^a A_{ijk}^a) \omega_j \otimes \omega_k .$$

Consequently

$$\tag{3.4} \tfrac{1}{2}\Delta \|f_*\|^2 = \sum_{i,j,a} (A_{ij}^a A_{ij}^a + A_i^a A_{ijj}^a) .$$

From (3.1) and (3.3), we get

$$\begin{aligned} \sum_j DA_{ij}^a \wedge \omega_j &= \sum_{j,k} A_{ijk}^a \omega_k \wedge \omega_j \\ &= d\Big(\sum_k A_{ik}^a \omega_k\Big) + \sum_k \Big(\sum_j A_{kj}^a \omega_j\Big) \wedge \omega_{ik} - \sum_b \Big(\sum_j A_{ij}^b \omega_j\Big) \wedge \omega_{ba}^* \\ &= -\tfrac{1}{2} \sum_{j,k,l} A_j^a R_{jikl} \omega_k \wedge \omega_l - \tfrac{1}{2} \sum_{b,c,d} A_i^b R_{bacd}^* \omega_c^* \wedge \omega_d^* \\ &= -\tfrac{1}{2} \sum_{k,l} \Big[\sum_j A_j^a R_{jikl} + \sum_{b,c,d} R_{bacd}^* A_i^b A_k^c A_l^d\Big] \omega_k \wedge \omega_l , \end{aligned}$$

which implies

$$\tag{3.5} A_{ijk}^a - A_{ikj}^a = - \sum_l A_l^a R_{likj} - \sum_{b,c,d} A_i^b A_k^c A_j^d R_{bacd}^* .$$

In (3.4)

$$\tag{3.6} \begin{aligned} &\sum_{a,i,j} (A_{ij}^a A_{ij}^a + A_i^a A_{ijj}^a) \\ &\quad = \sum_{a,i,j} (A_{ij}^a)^2 + \sum_{a,i,j} A_i^a (A_{ijj}^a - A_{jji}^a) + \sum_{a,i,j} A_i^a A_{jji}^a . \end{aligned}$$

Observing that $A_{ijk}^a = A_{jik}^a$ and taking into account (3.5) and (3.6), we can write the formula (3.4) for the Laplacian as

$$\tag{3.7} \begin{aligned} \tfrac{1}{2}\Delta \|f_*\|^2 = &\sum_{a,i,j} (A_{ij}^a)^2 + \sum_{a,i,j} R_{ij} A_i^a A_j^a \\ &- \sum_{\substack{a,b,c,d\\ i,j}} R_{abcd}^* A_i^a A_j^b A_i^c A_j^d + \sum_{a,i,j} A_i^a A_{jji}^a . \end{aligned}$$

If f is harmonic the last term in (3.7) vanishes.

4. Harmonic mappings of bounded dilatation

Let $A^a = (A_1^a, \cdots, A_m^a)$ and $A_i = (A_i^1, \cdots, A_i^n)$ be local vector fields in M and N, respectively. Then locally

$$\sum_{a=1}^{n} \|A^a\|^2 = \sum_{i=1}^{m} \|A_i\|^2 = \|f_*\|^2 .$$

If there are constants C_1 and C_2 such that

$$C_1 \leq \text{the sectional curvature of } M \leq C_2 ,$$

then at x we have

$$(m-1) C_1 \|f_*\|^2 \leq \sum R_{ij} A_i^a A_j^a \leq (m-1) C_2 \|f_*\|^2 , \tag{4.1}$$

where $\|f_*\|^2 = \Sigma(A_i^a)^2$. Similarly, if the sectional curvatures of N at $f(x)$ are bounded above by a constant C, then

$$\sum R^*_{abcd} A_i^a A_j^b A_i^c A_j^d \leq 2C \| \wedge^2 f_* \|^2 . \tag{4.2}$$

Theorem 4.1. *Let M and N be Riemannian manifolds of dimensions m and n respectively, and let $f: M \to N$ be a harmonic mapping of bounded dilatation (of order K). Then*

$$B \|f_*\|^2 \leq \frac{m-1}{2} k^2 K^4 A , \tag{4.3}$$

if $\|f_*\|^2$ attains a maximum at $x \in M$,

(a) *the sectional curvatures of M at x are bounded below by a nonpositive constant $-A$, or M is an Einstein manifold with the scalar curvature R at x satisfying $R \geq -m(m-1)A$, and*

(b) *the sectional curvatures of N at $f(x)$ are bounded above by a nonpositive constant $-B$.*

Proof. Since $\|f_*\|$ attains its maximum at x, $\Delta_x \|f_*\|^2 \leq 0$. Applying (3.7) we have

$$- \sum R^*_{abcd} A_i^a A_j^b A_i^c A_j^d \leq - \sum R_{ij} A_i^a A_j^a \tag{4.4}$$

at x. Condition (a) together with (4.1) gives

$$- \sum R_{ij} A_i^a A_j^a \leq (m-1) A \|f_*\|_x^2 . \tag{4.5}$$

Similarly, condition (b) and (4.2) imply

$$2B \| \wedge^2 f_* \|_x^2 \leq - \sum R^*_{abcd} A_i^a A_j^b A_i^c A_j^d . \tag{4.6}$$

From (4.4), (4.5) and (4.6) we obtain

$$2B \| \wedge^2 f_* \|_x^2 \leq (m - 1)A \| f_* \|_x^2 .$$

Finally, from Lemma 2.2 it follows that

$$B \| f_* \|_x^2 \leq \tfrac{1}{2}(m - 1)k^2K^4A , \tag{4.7}$$

which proves the theorem.

Corollary 4.1. *If M is locally flat and the sectional curvatures of N are bounded above by a negative constant $-B$, then either $\|f_*\|$ does not attain its maximum or f is a constant mapping.*

The following generalizes Theorem 5.3 in [3].

Corollary 4.2. *Let $f: M \to N$ be a harmonic mapping of bounded dilatation of order K with the function $\|f_*\|$ attaining its maximum on M. If*

(a) *the sectional curvatures of M are bounded below by a nonpositive constant $-A$, or M is an Einstein manifold with scalar curvature $\geq -m(m - 1)A$, and*

(b) *the sectional curvatures of N are bounded above by a negative constant $-B$, then*

$$\| \wedge^p f_* \|^{2/p} \leq k\binom{k}{p}^{1/p} \frac{m-1}{2} \frac{A}{B} K^4 , \qquad 1 \leq p \leq k.$$

Proof. Since (4.7) holds at every point of M, the result follows from (2.1).

Corollary 4.3. *Under the assumptions of Corollary 4.2, if $B \geq \frac{1}{2}(m - 1)k^2K^4A$ and M is connected, then the mapping f is distance decreasing. If $m = n$ and $B \geq \frac{1}{2}n(n - 1)K^4A$, then f is volume decreasing.*

Proof. From (4.7) we get

$$\| f_*(X) \|^2 \leq \frac{m-1}{2} k^2K^4 \frac{A}{B} \| X \|^2 .$$

Corollary 4.4. *Let M be a compact locally flat Riemannian manifold, N a Riemannian manifold of nonpositive constant curvature, and $f: M \to N$ a nonconstant harmonic mapping. Then N is locally flat.*

Corollary 4.4 is well known (see [1], [5]).

Proof. Since M is compact the inequality (4.7) holds at some point x. Hence, since f is not constant, $A = 0$ implies $B = 0$.

5. Generalizations of the Schwarz-Ahlfors lemma, Liouville's theorem and the little Picard theorem

Let $d\tilde{s}^2$ be a Riemannian metric of M conformally related to ds^2. Then there is a function $p > 0$ on M such that $d\tilde{s}^2 = p^2ds^2$. In the sequel, the elements of M referred to $d\tilde{s}^2$ will be distinguished with a tilda. The notation otherwise being as above, we have

$$(5.1)\qquad \tilde{A}_i^a = qA_i^a\ ,\quad \tilde{\omega}_i = p\omega_i\ ,\quad \tilde{\omega}_{ij} = \omega_{ij} + p_i\omega_j - p_j\omega_i\ ,$$

where $q = p^{-1}$, $dp = \Sigma p_i\tilde{\omega}_i$, $dq = \Sigma q_i\tilde{\omega}_i$ and $pq_i = -qp_i$. From (3.7) it follows that the Laplacian $\tilde{\Delta}$ of $\tilde{u} = \Sigma(\tilde{A}_i^a)^2$ with respect to $d\tilde{s}^2$ is

$$(5.2)\quad \tfrac{1}{2}\tilde{\Delta}\tilde{u} = \sum(\tilde{A}_{ij}^a)^2 + \sum \tilde{R}_{ij}\tilde{A}_i^a\tilde{A}_j^a - \sum R^*_{abcd}\tilde{A}_i^a\tilde{A}_j^b\tilde{A}_i^c\tilde{A}_j^d + \sum \tilde{A}_i^a\tilde{A}_{jji}^a\ .$$

By (3.2) and (3.3) we obtain

$$(5.3)\qquad \sum_k\Big(\sum_j \tilde{A}_{jjk}^a\Big)\tilde{\omega}_k = d\Big(\sum_j \tilde{A}_{jj}^a\Big) + \sum_b\Big(\sum_j \tilde{A}_{jj}^b\Big)\omega^*_{ba}\ .$$

On the other hand, (3.2), (3.3) and (5.1) imply

$$(5.4)\qquad \tilde{A}_{jj}^a = 2A_j^aq_j + q^2A_{jj}^a - \sum_k A_k^aq_k\ ,\qquad j\text{: not summed.}$$

If f is harmonic with respect to ds^2, then

$$(5.5)\qquad \sum_j \tilde{A}_{jj}^a = (2-m)\sum_k A_k^aq_k\ .$$

Substituting (5.5) into (5.3) we get

$$(5.6)\qquad \sum_j \tilde{A}_{jjk}^a = (2-m)q\sum_j(A_j^aq_{jk} + q_jA_{jk}^a)\ ,$$

where q_{jk} is defined by

$$dq_k + \sum_j q_j\omega_{jk} = \sum_j q_{kj}\omega_j\ ,\qquad q_{jk} = q_{kj}\ .$$

By (5.6), the last term in (5.2) becomes

$$(5.7)\qquad \sum_{a,i,j}\tilde{A}_i^a\tilde{A}_{jji}^a = (2-m)q^2\sum_{a,i,j}(A_i^aA_j^aq_{ji} + A_i^aA_{ji}^aq_j)\ .$$

If $\tilde{u}$ attains a maximum at $x\in M$, then

$$\sum A_i^aA_{ji}^a = p_j\sum(A_i^a)^2$$

at x. Formula (5.7) then becomes

$$(5.8)\qquad \sum_{a,i,j}\tilde{A}_i^a\tilde{A}_{jji}^a = (m-2)q^2\sum_{a,i,j}A_i^aA_j^a(Q\delta_{ij} - q_{ij})\ ,$$

where $Q = \sum_i (pq_i)^2$.

From (5.2) and (5.8) the following lemma is immediate.

Lemma 5.1. *Let f be harmonic with respect to (ds^2, ds^{*2}), and let $\tilde{u}$ attain its maximum at $x \in M$. If the symmetric matrix function*

$$X_{ij} = Q\delta_{ij} - q_{ij}$$

is positive semidefinite on M, then

$$-\sum R^*_{abcd}\tilde{A}^a_i\tilde{A}^b_j\tilde{A}^c_i\tilde{A}^d_j \leq -\sum \tilde{R}_{ij}\tilde{A}^a_i\tilde{A}^a_j$$

at x.

Theorem 5.1. *Let B^m be the m-dimensional unit open ball with the metric $ds^2 = 4A^{-1}(1-r^2)^{-2}\Sigma dx_i^2$ of constant negative curvature $-A$, and let N be an n-dimensional Riemannian manifold with sectional curvatures bounded above by a negative constant $-B$. If $f\colon B^m \to N$ is a harmonic mapping of bounded dilatation of order K, then*

$$\| \wedge^p f_* \|^{2/p} \leq k\binom{k}{p}^{1/p}\frac{m-1}{2}\frac{A}{B}K^4\,, \qquad 1 \leq p \leq k. \tag{5.9}$$

Proof. Let B_α be the open ball of radius $\alpha\,(<1)$. In B_α we take the metric $d\tilde{s}^2 = 4A^{-1}\alpha^2(\alpha^2 - r^2)^{-2}\Sigma dx_i^2$ with constant curvature $-A$. Then $d\tilde{s}^2 = p^2ds^2$ in B_α, where $p = \alpha(1-r^2)/(\alpha^2 - r^2)$ and $r^2 = \Sigma x_i^2$. The matrix X_{ij} is then given by

$$X_{ij} = \frac{A(1-\alpha^2)(\alpha^2 - r^2)(1+r^2)}{2\alpha^2(1-r^2)^2}\delta_{ij} + \frac{A(1-r^2)^2}{\alpha^2(\alpha^2-r^2)^2}(r^2\delta_{ij} - x_ix_j)\,.$$

Clearly, X_{ij} is positive semidefinite. The function

$$\tilde{u} = \sum (\tilde{A}^a_i)^2 = \left[\frac{\alpha^2 - r^2}{\alpha(1-r^2)}\right]^2 \sum (A^a_i)^2$$

attains its maximum on the closure $\bar{B}_\alpha$ of B_α. But $\tilde{u}$ vanishes on the boundary of $\bar{B}_\alpha$. Hence it attains its maximum at a point $x \in B_\alpha$. Applying Lemma 5.1 we get $-\Sigma R^*_{abcd}\tilde{A}^a_i\tilde{A}^b_j\tilde{A}^c_i\tilde{A}^d_j \leq (m-1)A\tilde{u}$, for $\tilde{R}_{ij} = -(m-1)A\delta_{ij}$. Let $\| \wedge^p f_* \|_{(\alpha)}$ denote the norm of $\wedge^p f_*$ with respect to $d\tilde{s}^2$. Then, as in the proof of Corollary 4.2,

$$2B \| \wedge^2 f_* \|^2_{(\alpha)} \leq (m-1)A \, \|f_*\|^4_{\alpha}$$

at x. Applying Lemma 2.2 gives

$$\|f_*\|^2_{(\alpha)} \leq \frac{m-1}{2}k^2\frac{A}{B}K^4$$

everywhere on B_α. Since the preceding inequality holds for every α, and $\lim_{\alpha\to 1} \|f_*\|^2_{(\alpha)} = \|f_*\|^2$, we conclude that

$$\|f_*\|^2 \leq \frac{m-1}{2}k^2\frac{A}{B}K^4\,.$$

Corollary 5.1. *Under the conditions in Theorem* 5.1, *if* $B \geq \frac{1}{2}(m-1)k^2AK^4$, *the mapping* f *is distance decreasing.*

In the case where $M = E^m$ with the standard flat metric, Corollary 4.1 can be improved as follows.

Theorem 5.2. *Let* N *be an n-dimensional Riemannian manifold with negative sectional curvature bounded away from zero, and let* $f\colon E^m \to N$ *be a harmonic mapping of bounded dilatation. Then* f *is a constant mapping.*

Proof. Let B_α be the open ball of radius α with metric $d\tilde{s}^2 = \alpha^4(\alpha^2 - r^2)^{-2}\Sigma dx_i^2$. Then $d\tilde{s}^2 = p^2\Sigma dx_i^2$ where $p = \alpha^2/(\alpha^2 - r^2)$. In this case,

$$X_{ij} = \frac{2(\alpha^2 - r^2)}{\alpha^4}\delta_{ij} + \frac{4}{\alpha^4}(r^2\delta_{ij} - x_ix_j) ,$$

so it is also positive semidefinite. Since the function $\tilde{u} = \|f_*\|^2_{(\alpha)} = q^2\Sigma(A_i^a)^2$ attains its maximum on $\bar{B}_\alpha$ and vanishes on the boundary of B_α, it must attain its maximum in B_α. Since the sectional curvature of N is bounded above by $-\varepsilon$ for some constant $\varepsilon > 0$, from the inequality (4.7) it follows that

$$\varepsilon \|f_*\|^2_{(\alpha)} \leq 2\alpha^{-2}(m-1)k^2K^4 .$$

Hence $\|f_*\|^2 = \lim_{\alpha \to \infty} \|f_*\|^2_{(\alpha)} = 0$.

If $\pi : S \to M$ is a Riemannian covering we have easily

Lemma 5.2. *Let* $f\colon M \to N$ *be a* C^∞ *mapping and* $\bar{f} = f \circ \pi$. *Then*

$$\| \wedge^p \bar{f}_* \|_x = \| \wedge^p f_* \|_{\pi(x)} , \qquad x \in S .$$

If M is a complete connected Riemannian manifold of constant curvature c, then its universal covering space is

$$S^m \quad \text{for} \quad c > 0 , \quad E^m \quad \text{for} \quad c = 0 \quad \text{and} \quad B^m \quad \text{for} \quad c < 0 ,$$

where S^m is the m-sphere of constant curvature c (> 0), and B^m is the unit open m-ball with the metric $ds^2 = -4c^{-1}(1 - r^2)^{-2}\Sigma dx_i^2$ of constant curvature c (< 0).

Hence by Proposition 4.1 of [3], Theorems 5.1 and 5.2 and Lemma 5.2 above, we get

Theorem 5.3. *Let* M *be a complete connected Riemannian manifold of positive constant curvature and let* N *be a manifold with nonpositive sectional curvature. Then a harmonic mapping from* M *into* N *is a constant mapping.*

This fact is well known [1].

Theorem 5.4. *Let* M *be a complete connected Riemannian manifold of constant negative curvature* $-A$ *and let* N *be a Riemannian manifold whose sectional curvatures are bounded above by a negative constant* $-B$. *If* $f\colon M \to N$ *is a harmonic mapping of bounded dilatation of order* K, *then the inequality* (5.9) *is satisfied.*

Thus, if $B \geq \frac{1}{2}(m - 1)k^2K^4A$, the mapping f is distance decreasing. In the equidimensional case, if $B \geq \frac{1}{2}n(n - 1)K^4A$, f is volume decreasing.

Theorem 5.5. *Let M be a complete connected locally flat Riemannian manifold and let N be a Riemannian manifold with negative sectional curvature bounded away from zero. Then a harmonic mapping of bounded dilatation* $f: M \to N$ *is a constant mapping.*

Theorem 5.5 generalizes Liouville's theorem and the little Picard theorem. For, in the first case, a bounded domain in the complex plane C is contained in a disc which has constant negative curvature with respect to the Poincaré metric, and in the latter case, C — {2 points} carries a Kaehler metric of negative curvature bounded away from zero.

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